Problem 1.10

*Perpendicular unit vectors**
Given vector $\mathbf{A} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$,

(a) find a unit vector $\mathbf{B}$ that lies in the x–y plane and is perpendicular to $\mathbf{A}$.

(b) find a unit vector $\hat{\mathbf{C}}$ that is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$.

(c) Show that $\mathbf{A}$ is perpendicular to the plane defined by $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$.

Solution

Part (a)

Taking the cross product of $\mathbf{A}$ and a unit vector in the z direction will give us another vector that is perpendicular to both of them. That is, there will be no component in the z direction.

$$\mathbf{A} \times \hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 4 & -4 \\ 0 & 0 & 1 \end{vmatrix} = 4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$$

The unit vector $\hat{\mathbf{B}}$ is obtained by dividing this vector by its magnitude.

$$\hat{\mathbf{B}} = \frac{4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}}{\sqrt{4^2 + (-3)^2}}$$

Therefore,

$$\hat{\mathbf{B}} = \frac{1}{5}(4\hat{\mathbf{i}} - 3\hat{\mathbf{j}}).$$

Part (b)

Taking the cross product of $\mathbf{A}$ and $\mathbf{B}$ will give us a vector perpendicular to both of them.

$$\mathbf{A} \times \hat{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 4 & -4 \\ \frac{4}{5} & -\frac{3}{5} & 0 \end{vmatrix} = -\frac{12}{5}\hat{\mathbf{i}} - \frac{16}{5}\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

The unit vector $\hat{\mathbf{C}}$ is obtained by dividing this vector by its magnitude.

$$\hat{\mathbf{C}} = \frac{-\frac{12}{5}\hat{\mathbf{i}} - \frac{16}{5}\hat{\mathbf{j}} - 5\hat{\mathbf{k}}}{\sqrt{\left(-\frac{12}{5}\right)^2 + \left(-\frac{16}{5}\right)^2 + (-5)^2}}$$

Therefore,

$$\hat{\mathbf{C}} = -\frac{1}{\sqrt{41}}\left(\frac{12}{5}\hat{\mathbf{i}} + \frac{16}{5}\hat{\mathbf{j}} + 5\hat{\mathbf{k}}\right).$$

Part (c)

We can show that $\mathbf{A}$ is perpendicular to the plane by showing it is parallel to the vector normal to the plane that we obtain by taking the cross product.

$$\hat{\mathbf{B}} \times \hat{\mathbf{C}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{4}{5} & -\frac{3}{5} & 0 \\ -\frac{12}{5\sqrt{41}} & -\frac{16}{5\sqrt{41}} & -\frac{5}{\sqrt{41}} \end{vmatrix} = \frac{3}{\sqrt{41}}\hat{\mathbf{i}} + \frac{4}{\sqrt{41}}\hat{\mathbf{j}} - \frac{4}{\sqrt{41}}\hat{\mathbf{k}}$$

Since

$$\mathbf{A} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}} = \sqrt{41}\left(\frac{3}{\sqrt{41}}\hat{\mathbf{i}} + \frac{4}{\sqrt{41}}\hat{\mathbf{j}} - \frac{4}{\sqrt{41}}\hat{\mathbf{k}}\right) = \sqrt{41}(\hat{\mathbf{B}} \times \hat{\mathbf{C}}),$$

$\mathbf{A}$ is a vector in the same direction as $\hat{\mathbf{B}} \times \hat{\mathbf{C}}$. The only difference between them is that the magnitude of $\mathbf{A}$ is $\sqrt{41}$ times larger than that of $\hat{\mathbf{B}} \times \hat{\mathbf{C}}$. Because $\hat{\mathbf{B}} \times \hat{\mathbf{C}}$ is normal to the plane by virtue of the cross product, $\mathbf{A}$ is as well.